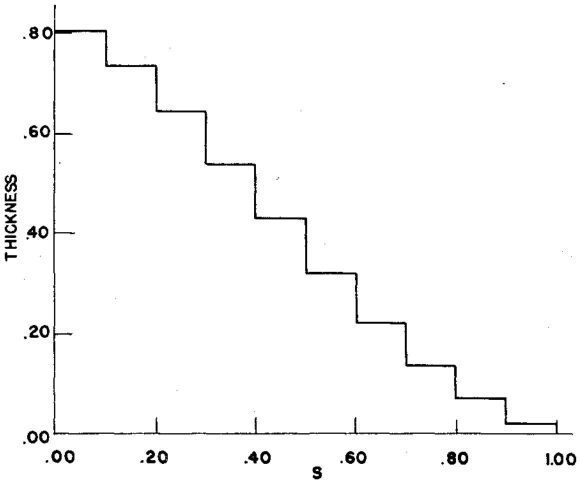

Fig. 3 Optimal thickness distribution with $\lambda_e < \lambda_I$.

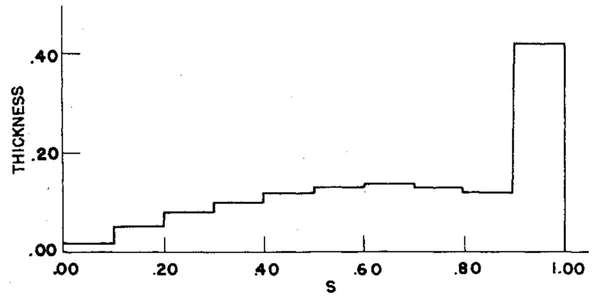

Fig. 4 Optimal thickness distribution with $\lambda_e > \lambda_I$.

algorithm designed by P. Rizzi.[6] This distribution has a first natural frequency that is less than the excitation frequency, indicating that this solution is in a feasible region unconnected from the feasible region associated with the distribution shown in Fig. 3. The thickness distribution of Fig. 4 is quite odd in that it has the minimum value specified at the root and has a large thickness concentrated near the tip. By studying the response of this structure, it can be shown that the effect of the large thickness near the tip is to create an inertial force that counteracts the applied load by acting out of phase with it. This tends to relieve the stresses throughout the structure.

The weight of the structures can be considered proportional to a factor equal to

$$\sum_{i=1}^{10} t_i/10$$

For the design of Fig. 3, this factor is 0.3932, while for Fig. 4, it is 0.1311.

Conclusions

This Note points out a novel and complicating feature in the study of structural optimization with harmonic or other dynamic excitation. In addition, an example is presented where the least weight solution is a design that is not intuitively reasonable. A number of questions relevant to the phenomenon of disjoint design spaces remain. Chief among these are the determination of the number of local optima that can exist for a given example and the development of an efficient method for seeking the various optima.

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Analytical Model for a Two-Phase Vacuum Plume

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Nomenclature

B	= constant
I_{sp}	= motor specific impulse
k	= constant in particle density equation ($k = 2.5$)
$\dot{m}$	= motor total mass flow rate
p	= pressure
r	= radial distance from exit plane
r^*	= nozzle throat radius
V	= velocity
$\bar{v}_p$	= average particle velocity
V_ℓ	= thermodynamic limiting velocity for the gas
ρ	= density
ξ	= nondimensional angular coordinate, $\xi = \phi/\phi_m$
ϕ	= azimuthal angle
ϕ_m	= limiting particle streamline angle
λ	= constant in gas density equation
ϵ	= nozzle exit area ratio
x_p	= mass fraction of solid particles

Subscripts

p	= particle
g	= gas
o	= chamber conditions

I. Introduction

THIS Note describes an approximate analytical model for the two-phase (gas/solid particle) vacuum plume from a solid propellant rocket motor. It is applicable to motors with thrust levels from a few hundred to several thousand pounds utilizing propellants containing the nominal (30% by weight) mass fraction of aluminum oxide particles in the exhaust products. The model provides a simple, analytical description of the undisturbed plume structure valid far from the exit plane (more than about 10 exit radii) which has been found useful in assessing plume impingement effects. Although essentially exact flowfield solutions for these plumes can be obtained from numerical codes, the approximate analytical model has proved valuable in both preliminary and more detailed analyses. This is particularly true for erosion and contamination estimates for which the uncertainty in the plume/surface interaction mechanisms

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make more precise flowfield predictions of marginal additional benefit.

The present model follows Hill and Draper's[1] technique of fitting an approximate analytical form to detailed numerical method-of-characteristics (MOC) plume solutions. Both the gas and particle plumes are described as source-like with densities that diminish according to an inverse square law with distance from the nozzle exit. Pearce[2] applied this technique previously to the particle mass flux. The present Note extends this description to include the gas mass flux (which is different from that in a purely gaseous plume) and the gas and particle momentum and kinetic energy fluxes. These additional quantities are needed for erosion, pressure, and heating estimates.

II. Derivation

The plume is described by the spherical coordinate system in Fig. 1 with the origin at the nozzle exit. It was shown[2] that the particle mass flux at any location r, $\phi(\phi \leq \phi_m)$ in the plume is approximated by

$$(\rho V)_p = \frac{2k\, x_p \dot{m} \xi \exp(-k^2\, \xi^4)}{\pi^{3/2} \phi_m r^2 \sin(\xi\, \phi_m)} \tag{1}$$

It was found that this functional form also approximates the particle momentum flux when a mean particle velocity $\bar{v}_p$ (weighted with respect to particle size and mass distribution) can be assigned that is essentially independent of position in the plume. The particle momentum flux is then given by

$$(\rho V^2)_p = (\rho V)_p \bar{v}_p \tag{2}$$

The angular dependence of the gas density was found to be adequately approximated by an exponential decay of the form $\exp[-\lambda(1-\cos\phi)]$, $\phi < \pi$. It is assumed that the gas has achieved its thermodynamic limiting velocity V_ℓ. (This approximation is poorer for the two-phase plume than for a purely gaseous plume because the hot particles act to maintain the gas temperature at a higher level such that $V_g < V_\ell$ except at very large distances.) When these expressions for the particle and gas mass and momentum fluxes are used in the overall mass and momentum conservation equations for the nozzle, the following specification of the gaseous plume is obtained

$$(\rho V)_g = \frac{(1-x_p)\dot{m}\lambda \exp[-\lambda(1-\cos\phi)]}{2\pi r^2 (1-e^{-\lambda})} \tag{3}$$

$$(\rho V^2)_g = (\rho V)_g\, V_\ell \tag{4}$$

λ is a constant given implicitly by

$$\frac{1}{1-e^{-\lambda}} - \frac{1}{\lambda} = \frac{\dfrac{I_{sp}}{V_\ell} - x_p \dfrac{\bar{v}_p}{V_\ell}\left(1 - \dfrac{\phi_m{}^2}{2k\pi^{1/2}}\right)^{\dagger}}{1-x_p} \tag{5}$$

The flux of kinetic energy, for either gas or particles, given by

$$\tfrac{1}{2}\rho V^3 = \tfrac{1}{2}(\rho V^2)^2/\rho V \tag{6}$$

No attempt was made to prescribe the flux of particle thermal energy because the particles cool predominantly by thermal radiation in the plume. Near the nozzle exit, the kinetic and thermal energy flux of the particles are comparable, but as the particles cool and accelerate in the plume, the total particle energy flux becomes predominantly kinetic.

Equations (1-6) define the plume model. The parameters needed for its use are the chamber conditions, mass flow rate, specific impulse, particle mass fraction, and in addition, the three parameters k, ϕ_m, and $\bar{v}_p$. k is determined from the fit of Eq. (1) to the angular dependent particles mass and momentum flux distribution and a value $k=2.5$ is recommended. The limiting particle streamline angle ϕ_m is obtained from a two-phase MOC solution. For motors of the class considered here, $\phi_m = 25\text{-}30°$ is recommended. Similarly, the average particle velocity recommended for use in the model is $\bar{v}_p / V_\ell = 0.7$. Both ϕ_m and $\bar{v}_p$ depend on motor chamber pressure, area ratio, and nozzle size. The recommended values are representative of the class of motors with chamber pressures near 500 psia, area ratios of 20-30, thrust levels of a few thousand pounds, and particle mass fractions of about 30% in the exhaust. Motors which differ substantially from these conditions may require different values for ϕ_m and $\bar{v}_p$ in order for the model to apply with adequate accuracy. For example, the higher chamber pressure, smaller area ratio motors used to initially verify the particle mass flux model[2] had limiting streamline angles in the range $35 < \phi_m < 40°$. The essential difficulty is that the model requires the a priori specification of the parameters ϕ_m and $\bar{v}_p$ which can only be determined from the two-phase MOC solutions. This compromise is made in the attempt to realistically account for the effects of nonequilibrium between the gas and particle velocities in the nozzle and plume expansion, and nonuniform particle sizes on the structure of the farfield gas and particle plumes.

III. Verification

A summary of more detailed comparisons[3] between the predictions of the plume model and two-phase MOC solutions will be given here in order to verify and assess the accuracy of the model. In all cases the MOC solutions were obtained for the expansion of a thermally and calorically perfect gas.[4] However, the code does make a realistic specification of velocity and thermal nonequilibrium between the gas and a distribution of particle sizes.

Angular distributions of particle mass and momentum fluxes obtained for three motors from MOC solutions and the plume model are compared in Fig. 2. This comparison suggests that the particle mass and momentum fluxes have essentially the same angular dependence and that the analytical form Eq. (1) is an adequate approximation. A similar comparison for the gas density is given in Fig. 3. The analytical form is a good approximation, at least for $\phi < 40°$, within which about 50% of the gas mass and momentum are contained.

The decay with distance from the nozzle exit on the plume axis given by the MOC solutions and plume model are compared in Fig. 4. It is verified that both the gas and particle flows are source-like and that the plume model does give

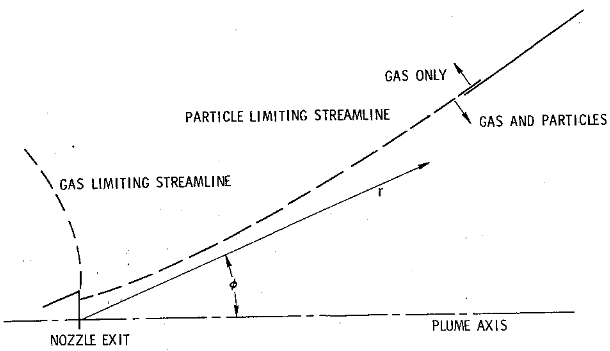

Fig. 1 Plume geometry and coordinate system.

†This relation follows from the momentum equation in which the pressure-area term was ignored as being negligible compared to the flux of momentum. The integral involving the particle momentum flux was evaluated approximately by expanding $\cos\phi$ in a Taylor series about $\phi = 0$ and ignoring terms beyond those of order ϕ^2.

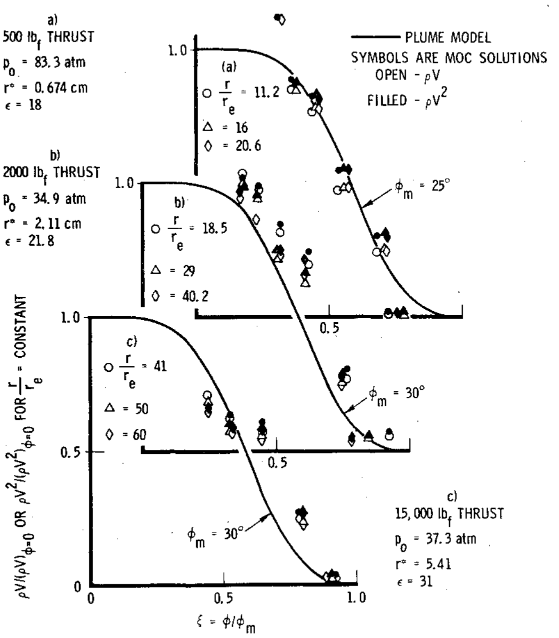

Fig. 2 Angular distribution of particle mass and momentum fluxes.

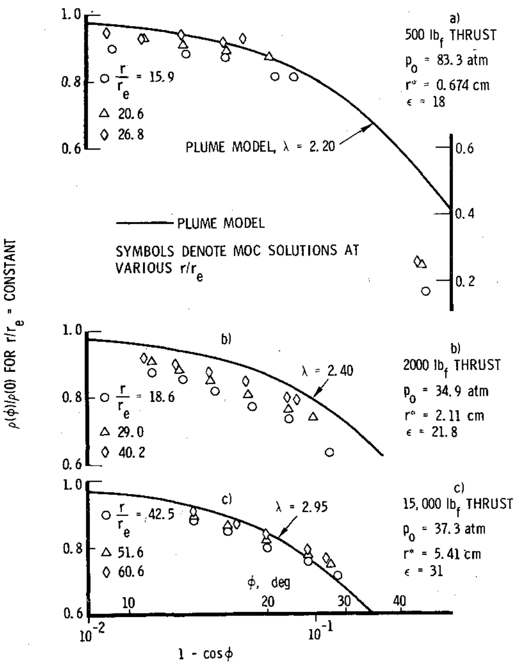

Fig. 3 Angular distribution of gas density.

adequate predictions for this class of motors. For the cases tested here, the plume model underpredicts the particle mass and momentum fluxes on the axis by 12-33%. The gas mass flux is essentially correctly predicted but the momentum flux is overpredicted by as much as 50%. This is a consequence of using V_ℓ for the gas velocity, which is most in error on the plume axis. The plume model predicts both the particle and gas mass and momentum fluxes with an accuracy comparable to those encountered in models for purely gaseous plumes,[1] which is judged to be acceptable in the applications for which the model is intended.

IV. Conclusions

The essential contribution of this model is that it provides a description of a two-phase plume accounting for a realistic particle size distribution and the velocity and thermal nonequilibrium in establishing the different spatial structure

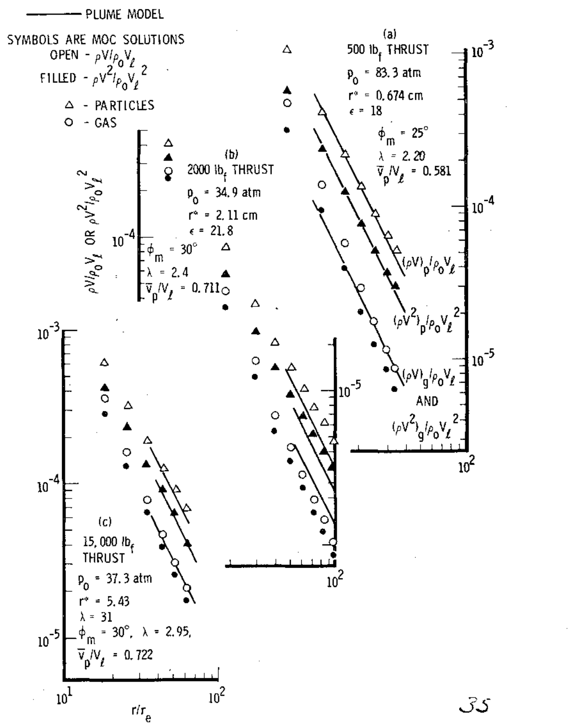

Fig. 4 Particle and gas mass and momentum fluxes on the plume axis.

for the gas and particle plumes far from the nozzle. It is an essential improvement over previous attempts to model a two-phase plume as an equilibrium mixture of gas and particles[5] which predicts total mass and momentum fluxes that are comparable to only those of the gas phase in the actual plume. Since the particle fluxes are as much as 3-5 times those of the gas (see Fig. 4) on the plume axis, the equilibrium mixture properties seriously underestimate the actual plume fluxes. Moreover, the equilibrium mixture does not account for the separate spatial structure of the particle plume. The present model, being based on actual coupled two-phase MOC calculations, is also more accurate for nominal metalized propellants than the approximate model proposed by Wang and Roberts.[6] They assume an exponential distribution of particle number density with angle of the form $\exp(-B\phi)$. A comparision of mass flux on the plume axis[3] shows that this assumption concentrates too much particle mass near the axis and overpredicts the mass flux by a factor 4.5 or more.

It is concluded that the approximate two-phase plume model proposed here provides a description of the gas and particle fluxes that is adequate for assessing plume impingement effects. It offers both superior accuracy and a more realistic description of the spatial structure of both the gas and particle plumes than previous approximate models.

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